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On flexural waves in cylindrically anisotropic elastic rods

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Abstract

A rod theory is developed for long waves in an elastic circular cylinder with cylindrical anisotropy. Detailed results for flexural waves and cylindrical orthotropy are given. The theory uses the method of Frobenius in the radial direction so that the equation of motion is satisfied exactly. Then, the equations arising from the lateral boundary condition are truncated properly, leading to a rod theory.

Keywords: Cylindrical orthotropy; Elastic waves; Method of Frobenius

1. Introduction

Classical beam theory says that the time-harmonic flexural vibrations of a bar are governed by

$$\frac{d^2}{dz^2}\left(EI\frac{d^2\psi}{dz^2}\right) - \varrho A\omega^2\psi = 0, \tag{1}$$

where $\psi(z)$ is the transverse displacement at position z along the bar, EI is the flexural rigidity, ϱ is the density, $A(z)$ is the cross-sectional area at z, and ω is the circular frequency. Derivations of (1) can be found in many textbooks on mechanical vibrations, for example, (Timoshenko, 1928, Section 40) or (Ginsberg, 2001, Section 7.1), see also (Rayleigh, 1945, Section187).

For cylindrical bars, we can seek solutions proportional to e^{ikz}, and then (1) gives $\varrho A\omega^2 = EIk^4$. In particular, for cylinders with circular cross-sections of radius a, we obtain

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$$\varrho(\omega a)^2 = \frac{1}{4}E(ka)^4, \tag{2}$$

where $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is Young's modulus, and λ and μ are the Lamé moduli. Eq. (2) is a dispersion relation. It predicts that flexural waves travel with speed c_0, where $c_0 = \frac{1}{2}ka(E/\varrho)^{1/2}$.

The calculations above raise a number of questions. When is the equation of motion, (1), justified? Eq. (1) is a one-dimensional approximation to the exact three-dimensional equations of motion; can this approximation be refined? Is the dispersion relation (2) correct? Can similar results be obtained for anisotropic bars?

Extensions of (1) to anisotropic cylinders were obtained by Voigt, Goens (1932) and others, and reviewed by Hearmon (1946). Thus, for a circular cylinder, Goens (1932) used the following system of equations,

$$\frac{1}{4}a^2E_A\psi^{\text{iv}} - \varrho\omega^2\psi + \frac{1}{4}a^2\gamma_A E_A\Theta''' = 0, \tag{3}$$

$$\mu_A\Theta' + \varrho\omega^2\Theta + \varepsilon_A\mu_A\psi''' = 0, \tag{4}$$

where $\Theta(z)$ is the angle of twist at z, and E_A, γ_A, μ_A and ε_A are given in terms of the elastic compliances; $E_A = E$, $\mu_A = \mu$ and $\gamma_A = \varepsilon_A = 0$ for isotropic solids. If we look for solutions in the form $\psi = \psi_0\,\mathrm{e}^{\mathrm{i}kz}$ and $\Theta = \Theta_0\mathrm{e}^{\mathrm{i}kz}$, we obtain a quadratic equation for ω as a function of k; one solution gives a pseudo-torsional wave and the other gives a pseudo-flexural wave; the latter satisfies $\varrho(\omega a)^2 = \frac{1}{4}E_A(ka)^4(1 + \varepsilon_A\gamma_A) + O((ka)^6)$ as $ka \to 0$, which agrees with (2) for isotropic solids. More generally, for orthotropic materials with symmetry planes aligned with the Cartesian xy, yz and zx planes, $\gamma_A = \varepsilon_A = 0$ and $E_A = E_3$, which is Young's modulus in the z-direction; for such materials, (3) and (4) decouple, and (2) is recovered with E replaced by E_3:

$$\varrho(\omega a)^2 = \frac{1}{4}E_3(ka)^4. \tag{5}$$

In the theory leading to (3) and (4), it is assumed that the cylinder is homogeneous with respect to the Cartesian form of Hooke's law; we shall be interested in cylindrical anisotropy, which is natural when cylindrical polar coordinates are used.

Eq. (2) is known to be correct in the following sense. The exact equations for wave propagation in an isotropic circular cylinder with a traction-free boundary can be solved exactly (using separation of variables in cylindrical polar coordinates, Bessel functions, and so on), see, for example, (Achenbach, 1973, Section 6.9) or (Graff, 1975, Section 8.2). The results show that an infinite number of different flexural modes exist. The lowest mode has a long-wavelength behaviour that agrees precisely with (2): the dispersion relation for the lowest mode can be written as $\omega a = F(ka)$ with $F(ka) \sim \frac{1}{2}(E/\varrho)^{1/2}(ka)^2$ as $ka \to 0$. For more on the comparison between exact and beam theories, see (Achenbach, 1973, Section 6.11.3) and (Love, 1927, Section 202).

For axisymmetric motions of an isotropic rod, Boström (2000) developed the following method. Solve the exact equations of motion for the displacement $\mathbf{u}(r,z)$ using power series in r, $\mathbf{u}(r,z) = \sum_{n=0}^{\infty} r^n\,\mathbf{u}_n(z)$, where r and z are cylindrical polar coordinates; this leads to a recursive structure in which $\mathbf{u}_n$ is determined in terms of $\mathbf{u}_m$ and derivatives of $\mathbf{u}_m$, with $m < n$. The lowest-order non-trivial terms remain undetermined until the lateral boundary condition is imposed. The exact boundary condition is written down; it involves a power series in the cross-sectional radius, a, which is assumed to be small. Truncation of this series leads to ordinary differential equations. Extension of this method to axisymmetric (non-cylindrical) bars and to anisotropic media is described in (Martin, 2004).

How should the series arising from the lateral boundary condition be truncated? Boström (2000) gives some discussion of this question; see also (Boström et al., 2001). However, the situation becomes more complicated for non-axisymmetric motions of anisotropic rods. Anisotropy leads to non-integer powers of r, so that the method of Frobenius is needed: various solutions of the equations of motion can be constructed. These solutions must be combined before the lateral boundary condition is imposed, as a consequence of non-axisymmetry.

We resolve the truncation issue here as follows. Any truncation will lead to an approximate dispersion relation. We ensure that this relation includes *all* terms that are $O((ka)^n)$ as $ka \to 0$, for $0 \leqslant n \leqslant N$ and a certain prescribed N. Thus, in this way, we obtain a dispersion relation for long flexural waves in an anisotropic rod. This relation reduces to (2) for isotropic rods. Extension to slender axisymmetric non-cylindrical bars will be described elsewhere.

2. Governing equations

Consider a cylindrically anisotropic elastic rod of infinite length and cross-sectional radius a. We are interested in the propagation of (long) flexural waves in such a rod. Let L be a typical axial wavelength; we assume that

$$\varepsilon = a/L \ll 1.$$

Let (r, θ, z) be cylindrical polar coordinates that have been made dimensionless using L. Thus, the rod is defined by $0 \leqslant r < \varepsilon$, $0 \leqslant \theta < 2\pi$ and $-\infty < z < \infty$.

The governing equation of motion is

$$\frac{\partial}{\partial r}\left(r\tilde{\mathbf{t}}_r\right) + \frac{\partial}{\partial \theta}\tilde{\mathbf{t}}_\theta + \mathbf{K}\tilde{\mathbf{t}}_\theta + r\frac{\partial}{\partial z}\tilde{\mathbf{t}}_z = \varrho L^2 r \frac{\partial^2 \tilde{\mathbf{u}}}{\partial t^2}, \tag{6}$$

where $\tilde{\mathbf{t}}_r = (\tau_{rr}, \tau_{r\theta}, \tau_{rz})^{\mathrm{T}}$, $\tilde{\mathbf{t}}_\theta = (\tau_{\theta r}, \tau_{\theta\theta}, \tau_{\theta z})^{\mathrm{T}}$, $\tilde{\mathbf{t}}_z = (\tau_{zr}, \tau_{z\theta}, \tau_{zz})^{\mathrm{T}}$,

$$\mathbf{K} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\mathbf{u}} = \begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \tag{7}$$

is the displacement vector, ϱ is the density, and τ_{ij} are the stress components. In what follows, we use a generalization of the matrix formulation of Ting (1996) for static problems. From (Ting, 1996), we have the following expressions for the traction vectors $\tilde{\mathbf{t}}_i$ in terms of $\tilde{\mathbf{u}}$:

$$\begin{aligned} \tilde{\mathbf{t}}_r &= \mathbf{Q}\frac{\partial}{\partial r}\tilde{\mathbf{u}} + \frac{1}{r}\mathbf{R}\left(\frac{\partial}{\partial \theta}\tilde{\mathbf{u}} + \mathbf{K}\tilde{\mathbf{u}}\right) + \mathbf{P}\frac{\partial}{\partial z}\tilde{\mathbf{u}}, \\ \tilde{\mathbf{t}}_\theta &= \mathbf{R}^{\mathrm{T}}\frac{\partial}{\partial r}\tilde{\mathbf{u}} + \frac{1}{r}\mathbf{T}\left(\frac{\partial}{\partial \theta}\tilde{\mathbf{u}} + \mathbf{K}\tilde{\mathbf{u}}\right) + \mathbf{S}\frac{\partial}{\partial z}\tilde{\mathbf{u}}, \\ \tilde{\mathbf{t}}_z &= \mathbf{P}^{\mathrm{T}}\frac{\partial}{\partial r}\tilde{\mathbf{u}} + \frac{1}{r}\mathbf{S}^{\mathrm{T}}\left(\frac{\partial}{\partial \theta}\tilde{\mathbf{u}} + \mathbf{K}\tilde{\mathbf{u}}\right) + \mathbf{M}\frac{\partial}{\partial z}\tilde{\mathbf{u}}. \end{aligned} \tag{8}$$

In these expressions,

$$\mathbf{Q} = \begin{pmatrix} C_{11} & C_{16} & C_{15} \\ C_{16} & C_{66} & C_{56} \\ C_{15} & C_{56} & C_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} C_{16} & C_{12} & C_{14} \\ C_{66} & C_{26} & C_{46} \\ C_{56} & C_{25} & C_{45} \end{pmatrix},$$

$$\mathbf{T}=\begin{pmatrix} C_{66} & C_{26} & C_{46} \\ C_{26} & C_{22} & C_{24} \\ C_{46} & C_{24} & C_{44} \end{pmatrix}, \quad \mathbf{P}=\begin{pmatrix} C_{15} & C_{14} & C_{13} \\ C_{56} & C_{46} & C_{36} \\ C_{55} & C_{45} & C_{35} \end{pmatrix},$$

$$\mathbf{M}=\begin{pmatrix} C_{55} & C_{45} & C_{35} \\ C_{45} & C_{44} & C_{34} \\ C_{35} & C_{34} & C_{33} \end{pmatrix}, \quad \mathbf{S}=\begin{pmatrix} C_{56} & C_{46} & C_{36} \\ C_{25} & C_{24} & C_{23} \\ C_{45} & C_{44} & C_{34} \end{pmatrix},$$

$\mathbf{R}^{\mathrm{T}}$ is the transpose of $\mathbf{R}$, and we have used the contracted notation $C_{\alpha\beta}$ for the elastic stiffnesses with $(1,2,3)=(r,\theta,z)$.

We look for time-harmonic solutions of (6) in the form

$$\tilde{\mathbf{u}}(r,\theta,z,t)=\mathrm{Re}_{\mathrm{i}}\{\mathbf{u}(r)\mathrm{e}^{\mathrm{j}m\theta}\mathrm{e}^{\mathrm{i}(\xi z-\omega t)}\}, \tag{9}$$

with similar expressions for $\tilde{\mathbf{t}}_i$. Here, i and j are non-interacting complex units, m is an integer, ω is the radian frequency and ξ is a dimensionless axial wavenumber. Use of $\mathrm{e}^{\mathrm{j}m\theta}$ rather than $\cos m\theta$ and $\sin m\theta$ allows us to retain matrix notation below; at the end of the calculation, we can take the real part with respect to j, or the imaginary part. We find that $\mathbf{u}(r)$ solves

$$r\mathbf{Q}(r\mathbf{u}')'+r(\mathbf{R}\mathbf{K}_m+\mathbf{K}_m\mathbf{R}^{\mathrm{T}})\mathbf{u}'+\mathrm{i}\xi r^2(\mathbf{P}+\mathbf{P}^{\mathrm{T}})\mathbf{u}'-\xi^2r^2\mathbf{M}\mathbf{u}+\mathrm{i}\xi r(\mathbf{P}+\mathbf{K}_m\mathbf{S}+\mathbf{S}^{\mathrm{T}}\mathbf{K}_m)\mathbf{u} \\ +\left\{\varrho(\omega L r)^2\mathbf{I}+\mathbf{K}_m\mathbf{T}\mathbf{K}_m\right\}\mathbf{u}=\mathbf{0}, \tag{10}$$

where $\mathbf{I}$ is the identity and $\mathbf{K}_m=\mathbf{K}+\mathrm{j}m\mathbf{I}$. From (8), we also have

$$\mathbf{t}_r=\mathbf{Q}\mathbf{u}'+r^{-1}\mathbf{R}\mathbf{K}_m\mathbf{u}+\mathrm{i}\xi\mathbf{P}\mathbf{u}. \tag{11}$$

For axisymmetric ($m=0$) motions, we recover the equations studied in (Martin and Berger, 2003; Martin, 2004). For two-dimensional motions independent of z, we recover the equations studied in (Martin and Berger, 2001). If we also put $m=0$ and $\omega=0$ (static), we obtain the equations solved by Ting (1996).

Setting $\mathbf{u}=(u,v,w)^{\mathrm{T}}$, (10) gives three coupled ordinary differential equations for the three components of $\mathbf{u}$. In general, these equations do not decouple.

The lateral boundary of the rod is free from tractions, whence

$$\mathbf{t}_r=\mathbf{0} \quad \text{on } r=\varepsilon, \quad -\infty<z<\infty. \tag{12}$$

3. The method of Frobenius

To solve (10), we use the method of Frobenius. This method has been used by many authors; see, for example, (Ohnabe and Nowinski, 1971; Chou and Achenbach, 1972; Markuš and Mead, 1995; Yuan and Hsieh, 1998; Martin and Berger, 2001; Shuvalov, 2003) and references therein. Thus, we write

$$\mathbf{u}(r)=\sum_{n=0}^{\infty}r^{n+\alpha}\mathbf{u}_n^{(m)}(\alpha). \tag{13}$$

Substitution in (10) gives

$$\mathbf{0}=\sum_{n=0}^{\infty}r^{n+\alpha}\mathbf{G}_n^{(m)}\mathbf{u}_n^{(m)}+\mathrm{i}\xi\sum_{n=1}^{\infty}r^{n+\alpha}\mathbf{A}_n^{(m)}\mathbf{u}_{n-1}^{(m)}+\sum_{n=2}^{\infty}r^{n+\alpha}\mathbf{B}\mathbf{u}_{n-2}^{(m)}, \tag{14}$$

where $\mathbf{B}=\varrho(\omega L)^2\mathbf{I}-\xi^2\mathbf{M}$,

$$\mathsf{G}_n^{(m)}(\alpha) = (n+\alpha)^2\mathsf{Q} + (n+\alpha)(\mathsf{R}\mathsf{K}_m + \mathsf{K}_m\mathsf{R}^{\mathrm{T}}) + \mathsf{K}_m\mathsf{T}\mathsf{K}_m,$$
$$\mathsf{A}_n^{(m)}(\alpha) = (n-1+\alpha)(\mathsf{P}+\mathsf{P}^{\mathrm{T}}) + \mathsf{P} + \mathsf{K}_m\mathsf{S} + \mathsf{S}^{\mathrm{T}}\mathsf{K}_m. \tag{15}$$

Notice that $\mathsf{G}_n^{(m)}(\alpha) = \mathsf{G}_0^{(m)}(n+\alpha)$ and $\mathsf{A}_n^{(m)}(\alpha) = \mathsf{A}_1^{(m)}(n-1+\alpha)$.

For (14) to be satisfied, we must first have

$$\mathsf{G}_0^{(m)}(\alpha)\,\mathbf{u}_0^{(m)}(\alpha) = \mathbf{0}. \tag{16}$$

The terms with $n = 1$ give

$$\mathsf{G}_1^{(m)}(\alpha)\,\mathbf{u}_1^{(m)}(\alpha) + \mathrm{i}\xi\mathsf{A}_1^{(m)}(\alpha)\,\mathbf{u}_0^{(m)}(\alpha) = \mathbf{0}. \tag{17}$$

Subsequent terms give

$$\mathsf{G}_n^{(m)}(\alpha)\mathbf{u}_n^{(m)} + \mathrm{i}\xi\mathsf{A}_n^{(m)}(\alpha)\mathbf{u}_{n-1}^{(m)} + \mathsf{B}\mathbf{u}_{n-2}^{(m)} = \mathbf{0}, \quad n \geqslant 2. \tag{18}$$

Eq. (16) has a non-trivial solution provided that

$$\det \mathsf{G}_0^{(m)}(\alpha) = 0. \tag{19}$$

This equation, the *indicial equation*, determines α; in general, there are six solutions. For each allowable α, (16) then determines the form of (the eigenvector) $\mathbf{u}_0$. We are usually interested in non-negative (real) α because we want $\mathbf{u}(r)$ to be bounded at $r = 0$. Also, as in the static case (Ting, 1996; Tarn, 2002), values of α with $0 < \alpha < 1$ give rise to singular stresses at $r = 0$; this is a consequence of the cylindrical-anisotropy model.

Once we have selected an allowable value for α, we can use (17) and (18) to determine $\mathbf{u}_1, \mathbf{u}_2, \ldots$ in terms of $\mathbf{u}_0$. Then, regardless of the choice of $\mathbf{u}_0$, the infinite series in (13) will give an exact solution of (10), assuming that the series converges. We can do this for each allowable α; note that $\mathbf{u}_0$ can be different for different values of α: $\mathbf{u}_0 = \mathbf{u}_0(\alpha)$. Then, we can form linear combinations of these solutions $\mathbf{u}$ (with respect to α). Finally, we can impose the lateral boundary condition (discussed next), in order to determine $\mathbf{u}_0(\alpha)$.

3.1. Lateral boundary condition

Substituting (13) in (11) gives $r\mathbf{t}_r(r) = \sum_{n=0}^{\infty} r^{n+\alpha}\mathbf{b}_n^{(m)}(\alpha)$, where

$$\mathbf{b}_0^{(m)}(\alpha) = (\alpha\mathsf{Q} + \mathsf{R}\mathsf{K}_m)\mathbf{u}_0^{(m)}(\alpha), \tag{20}$$

$$\mathbf{b}_n^{(m)}(\alpha) = \{(n+\alpha)\mathsf{Q} + \mathsf{R}\mathsf{K}_m\}\mathbf{u}_n^{(m)}(\alpha) + \mathrm{i}\xi\mathsf{P}\mathbf{u}_{n-1}^{(m)}(\alpha), \quad n \geqslant 1. \tag{21}$$

From the calculations above, we know (in principle, at least) how to express $\mathbf{u}_n^{(m)}$ in terms of $\mathbf{u}_l^{(m)}$ with $0 \leqslant l < n$; doing this ensures that the governing equation of motion is satisfied. Then, using a weighted sum over all allowable α, we write

$$\mathbf{u}(r) = \sum_{\alpha} \varepsilon^{-\alpha} \sum_{n=0}^{\infty} r^{n+\alpha}\mathbf{u}_n^{(m)}(\alpha). \tag{22}$$

The factors of $\varepsilon^{-\alpha}$ here are algebraically convenient; note that the coefficients $\mathbf{u}_n^{(m)}(\alpha)$ could depend on ε. The lateral boundary condition (12) on $r = \varepsilon$ gives

$$\sum_{\alpha}\sum_{n=0}^{\infty} \varepsilon^n \mathbf{b}_n^{(m)}(\alpha) = \mathbf{0}. \tag{23}$$

Then, our strategy is to truncate (23) in order to satisfy the lateral boundary condition, approximately. Our precise truncation strategy will be given later after we have discussed solutions of the indicial equation (19).

At this point, we simplify the calculations by considering materials with cylindrical orthotropy. This is a plausible model for wood, and includes isotropic materials as a special case. Calculations for axisymmetric motions, for which $m = 0$, are made in (Martin, 2004); this is the simplest situation. Here, we assume that $m \neq 0$, and are especially interested in flexural motions for which $m = 1$.

4. Cylindrical orthotropy

For materials with cylindrical orthotropy, there are nine non-trivial stiffnesses, namely C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}, C_{44}, C_{55} and C_{66}. The matrices $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{T}$, $\mathbf{P}$, $\mathbf{M}$ and $\mathbf{S}$ simplify to

$$\mathbf{Q} = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{66} & 0 \\ 0 & 0 & C_{55} \end{pmatrix}, \quad \mathbf{R} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{66} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{T} = \begin{pmatrix} C_{66} & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & C_{44} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & 0 & C_{13} \\ 0 & 0 & 0 \\ C_{55} & 0 & 0 \end{pmatrix},$$

$$\mathbf{M} = \begin{pmatrix} C_{55} & 0 & 0 \\ 0 & C_{44} & 0 \\ 0 & 0 & C_{33} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_{23} \\ 0 & C_{44} & 0 \end{pmatrix},$$

and the system (10) simplifies accordingly.

Isotropy is a special case of cylindrical orthotropy. For isotropic materials, $C_{11} = C_{22} = C_{33} = \lambda + 2\mu$, $C_{12} = C_{13} = C_{23} = \lambda$ and $C_{44} = C_{55} = C_{66} = \mu$, where λ and μ are the Lamé moduli. Exact solutions of (10) are well known for isotropic solids; see, for example, (Achenbach, 1973, Section 6.9) or (Graff, 1975, Section 8.2).

Elementary calculations give

$$\mathbf{G}_0^{(m)}(\alpha) = \begin{pmatrix} G_{11} & G_{12} & 0 \\ G_{21} & G_{22} & 0 \\ 0 & 0 & G_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{A}_1^{(m)}(\alpha) = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ A_{31} & A_{32} & 0 \end{pmatrix}, \tag{24}$$

where

$$G_{11} = \alpha^2 C_{11} - C_{22} - m^2 C_{66}, \quad G_{12} = \mathrm{j}m[\alpha(C_{12} + C_{66}) - C_{22} - C_{66}],$$
$$G_{21} = \mathrm{j}m[\alpha(C_{12} + C_{66}) + C_{22} + C_{66}], \quad G_{22} = (\alpha^2 - 1)C_{66} - m^2 C_{22},$$
$$G_{33} = \alpha^2 C_{55} - m^2 C_{44},$$
$$A_{13} = (\alpha + 1)C_{13} - C_{23} + \alpha C_{55}, \quad A_{31} = \alpha C_{13} + C_{23} + (\alpha + 1)C_{55}$$

and

$$A_{23} = A_{32} = \mathrm{j}m(C_{23} + C_{44}).$$

For cylindrical orthotropy, we can simplify (19) using (24) to give

$$\det \mathbf{G}_0^{(m)}(\alpha) = C_{66}^2(\alpha^2 C_{55} - m^2 C_{44})\Delta_0(\alpha; m) = 0, \tag{25}$$

where $\Delta_0(\alpha; m) = \alpha^4 c_1 + \alpha^2\{m^2(c_{12}^2 + 2c_{12} - c_1 c_2) - c_1 - c_2\} + (m^2 - 1)^2 c_2$, $c_1 = C_{11}/C_{66}$, $c_{12} = C_{12}/C_{66}$ and $c_2 = C_{22}/C_{66}$. The equation $\Delta_0(\alpha; m) = 0$ was investigated in (Martin and Berger, 2001, p. 1163). It is a quadratic in α^2. Hence, all six solutions of (25) can be found explicitly. We are interested in non-negative (real) solutions because we want displacements that are bounded at $r = 0$.

5. Flexural vibrations of a rod

In the remainder of the paper, we consider flexural vibrations only, for which $m = 1$. *Henceforth, we suppress the superscript* '(1)' to simplify notation. Then, (25) has three non-negative solutions; they are $\alpha = 0$, $\alpha = \tilde{\alpha}$ and $\alpha = \hat{\alpha}$, where $\hat{\alpha} = (C_{44}/C_{55})^{1/2}$ and

$$\tilde{\alpha} = \left\{[C_{11}C_{22} - C_{12}^2 + C_{66}(C_{11} - 2C_{12} + C_{22})]/(C_{11}C_{66})\right\}^{1/2}. \tag{26}$$

(For isotropic solids, we obtain $\hat{\alpha} = 1$ and $\tilde{\alpha} = 2$.) Thus, we have

$$\det \mathbf{G}_0(\alpha) = C_{11}C_{55}C_{66}\alpha^2(\alpha^2 - \tilde{\alpha}^2)(\alpha^2 - \hat{\alpha}^2).$$

Introduce the notation $\langle \mathbf{b} \rangle_i$ for the ith component of the vector $\mathbf{b}$, $i = 1, 2, 3$. Then, with $\mathbf{u} = (u, \mathrm{j}v, w)^{\mathrm{T}}$ (note the factor j, introduced for algebraic convenience), we obtain

$$\langle \mathbf{G}_0(\alpha)\mathbf{u} \rangle_1 = (\alpha^2 C_{11} - C_{22} - C_{66})u - [\alpha(C_{12} + C_{66}) - C_{22} - C_{66}]v,$$

$$\langle \mathbf{G}_0(\alpha)\mathbf{u} \rangle_2 = \mathrm{j}\{[\alpha(C_{12} + C_{66}) + C_{22} + C_{66}]u + [(\alpha^2 - 1)C_{66} - C_{22}]v\},$$

$$\langle \mathbf{G}_0(\alpha)\mathbf{u} \rangle_3 = (\alpha^2 C_{55} - C_{44})w,$$

$$\langle \mathbf{A}_1(\alpha)\mathbf{u} \rangle_1 = [(\alpha + 1)C_{13} - C_{23} + \alpha C_{55}]w,$$

$$\langle \mathbf{A}_1(\alpha)\mathbf{u} \rangle_2 = \mathrm{j}(C_{23} + C_{44})w,$$

$$\langle \mathbf{A}_1(\alpha)\mathbf{u} \rangle_3 = [\alpha C_{13} + C_{23} + (\alpha + 1)C_{55}]u - (C_{23} + C_{44})v.$$

From (20) and (21), with $\mathbf{u}_n = (u_n, \mathrm{j}v_n, w_n)^{\mathrm{T}}$, we obtain

$$\mathbf{b}_0(\alpha) = [(\alpha C_{11} + C_{12})u_0 - C_{12}v_0, \mathrm{j}C_{66}\{u_0 + (\alpha - 1)v_0\}, \alpha C_{55}w_0]^{\mathrm{T}} \tag{27}$$

and, for $n \geqslant 1$,

$$\langle \mathbf{b}_n(\alpha) \rangle_1 = [(n + \alpha)C_{11} + C_{12}]u_n - C_{12}v_n + \mathrm{i}\xi C_{13}w_{n-1}, \tag{28}$$

$$\langle \mathbf{b}_n(\alpha) \rangle_2 = \mathrm{j}C_{66}[u_n + (n + \alpha - 1)v_n], \tag{29}$$

$$\langle \mathbf{b}_n(\alpha) \rangle_3 = C_{55}[(n + \alpha)w_n + \mathrm{i}\xi u_{n-1}]. \tag{30}$$

Now, we shall truncate (23), arising from the lateral boundary condition, as follows:

$$\sum_{\alpha \geqslant 0}\left\{\mathbf{b}_0(\alpha) + \varepsilon\mathbf{b}_1(\alpha) + \varepsilon^2\mathbf{b}_2(\alpha)\right\} + \varepsilon^3\mathbf{b}_3(0) + \varepsilon^4\mathbf{b}_4(0) = \mathbf{0}. \tag{31}$$

Thus, we include two more terms corresponding to $\alpha = 0$ than those corresponding to $\alpha = \tilde{\alpha}$ and $\alpha = \hat{\alpha}$. This will ensure that we obtain a consistent approximation for long waves; see Appendix A. Evidently, more terms could be included in (31) if desired, but at the expense of further calculations.

In Appendix B, we give detailed solutions for $\alpha = 0$, $\alpha = \tilde{\alpha}$ and $\alpha = \hat{\alpha}$. Here, we summarise the results.

5.1. Solution for $\alpha = 0$

For $\alpha = 0$, (31) shows that we need $\mathbf{u}_n$ for $0 \leqslant n \leqslant 4$. Eq. (16) (with $m = 1$ and $\alpha = 0$) gives

$$\mathbf{u}_0 = (u_0, \mathrm{j}u_0, 0)^{\mathrm{T}}, \tag{32}$$

for some constant u_0. Further calculation gives $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ and $\mathbf{u}_4$ in terms of u_0. The corresponding tractions are given by

$$\mathbf{b}_0(0) = \mathbf{b}_1(0) = \mathbf{0},$$

$$\mathbf{b}_2(0) = u_0[\mathscr{C}_1\varrho(\omega L)^2 - \xi^2\mathscr{A}, \mathrm{j}\{\mathscr{C}_2\varrho(\omega L)^2 + \xi^2\mathscr{A}\}, 0]^{\mathrm{T}}, \tag{33}$$

$$\mathbf{b}_3(0) = u_0[0, 0, \mathrm{i}\xi\{\mathscr{C}_3\varrho(\omega L)^2 - \xi^2\mathscr{A}_3\}]^{\mathrm{T}}, \tag{34}$$

$$\langle\mathbf{b}_4(0)\rangle_1 = u_0\{\varrho^2(\omega L)^4\mathscr{H}_1 - \varrho(\omega L)^2\xi^2\mathscr{J}_1 + \xi^4\mathscr{L}_1\}, \tag{35}$$

$$\langle\mathbf{b}_4(0)\rangle_2 = \mathrm{j}u_0\{\varrho^2(\omega L)^4\mathscr{H}_2 - \varrho(\omega L)^2\xi^2\mathscr{J}_2 + \xi^4\mathscr{L}_2\} \tag{36}$$

and $\langle\mathbf{b}_4(0)\rangle_3 = 0$. All the constants appearing in (33)–(36) are defined in Appendix B. In particular, we note that

$$\mathscr{C}_1 + \mathscr{C}_2 = -1. \tag{37}$$

5.2. Solution for $\alpha = \tilde{\alpha}$

This solution has a similar form to that obtained when $\alpha = 0$. We find that

$$\mathbf{u}_0 = (\tilde{U}_0\tilde{v}_0, \mathrm{j}\tilde{v}_0, 0)^{\mathrm{T}}$$

for some constant $\tilde{v}_0$, where

$$\tilde{U}_0 = [\tilde{\alpha}(C_{12} + C_{66}) - C_{22} - C_{66}]/(\tilde{\alpha}^2C_{11} - C_{22} - C_{66}). \tag{38}$$

Further calculation gives $\mathbf{u}_1$ and $\mathbf{u}_2$. The corresponding tractions are given by

$$\mathbf{b}_0(\tilde{\alpha}) = \tilde{v}_0[\tilde{\mathscr{E}}, -\mathrm{j}\tilde{\mathscr{E}}, 0]^{\mathrm{T}}, \tag{39}$$

$$\mathbf{b}_1(\tilde{\alpha}) = \tilde{v}_0[0, 0, \mathrm{i}\xi\tilde{\mathscr{D}}_3]^{\mathrm{T}}, \tag{40}$$

$$\mathbf{b}_2(\tilde{\alpha}) = \tilde{v}_0[\tilde{\mathscr{C}}_1\varrho(\omega L)^2 - \xi^2\tilde{\mathscr{A}}_1, \mathrm{j}\{\tilde{\mathscr{C}}_2\varrho(\omega L)^2 - \xi^2\tilde{\mathscr{A}}_2\}, 0]^{\mathrm{T}}, \tag{41}$$

where all the constants appearing here are defined in Appendix B.

5.3. Solution for $\alpha = \hat{\alpha}$

In this case, we find that $\mathbf{u}_0 = (0, 0, \hat{w}_0)^{\mathrm{T}}$, where $\hat{w}_0$ is a constant. After calculating $\mathbf{u}_1$ and $\mathbf{u}_2$, we obtain expressions for the corresponding tractions:

$$\mathbf{b}_0(\hat{\alpha}) = \hat{w}_0[0, 0, \hat{\alpha}C_{55}]^{\mathrm{T}}, \tag{42}$$

$$\mathbf{b}_1(\hat{\alpha}) = \hat{w}_0[\mathrm{i}\xi\hat{\mathscr{D}}_1, \mathrm{ji}\xi\hat{\mathscr{D}}_2, 0]^{\mathrm{T}}, \tag{43}$$

$$\mathbf{b}_2(\hat{\alpha}) = \hat{w}_0[0, 0, \hat{\mathscr{C}}\varrho(\omega L)^2 - \xi^2\hat{\mathscr{A}}]^{\mathrm{T}}, \tag{44}$$

where $\hat{\mathscr{A}}$, $\hat{\mathscr{C}}$, $\hat{\mathscr{D}}_1$ and $\hat{\mathscr{D}}_2$ are defined in Appendix B. In particular, we have

$$\hat{\mathscr{D}}_1 + \hat{\mathscr{D}}_2 = -\hat{\alpha} C_{55}. \tag{45}$$

5.4. Isotropic limit

The solutions obtained in Section 5.2 when $\alpha = \tilde{\alpha}$ are well defined for isotropic media (for which $\tilde{\alpha} = 2$). Thus,

$$\tilde{U}_0 = \frac{\lambda - \mu}{3\lambda + 5\mu} = \frac{4v - 1}{5 - 4v},$$

where v is Poisson's ratio. Similarly,

$$\tilde{\mathscr{E}} = -\frac{4\mu}{5 - 4v}, \quad \tilde{\mathscr{D}}_3 = \frac{2\mu(1 + 2v)}{5 - 4v},$$

$$\tilde{\mathscr{A}}_1 = -\frac{2\mu(1 + v)}{3(5 - 4v)}, \quad \tilde{\mathscr{A}}_2 = -\frac{2\mu(2 - v)}{3(5 - 4v)},$$

$$\tilde{\mathscr{C}}_1 = \frac{1 - 6v + 4v^2}{6(1 - v)(5 - 4v)}, \quad \tilde{\mathscr{C}}_2 = \frac{12v - 7 - 4v^2}{6(1 - v)(5 - 4v)}.$$

However, those obtained in Section 5.1 (when $\alpha = 0$) and in Section 5.3 (when $\alpha = \hat{\alpha}$) are indeterminate. To obtain the isotropic limit in these two cases, we begin by specialising to tetragonal materials, for which $C_{11} = C_{22}$, $C_{13} = C_{23}$ and $C_{44} = C_{55}$; see Appendix B. Then, for $\alpha = 0$, we find that

$$\mathscr{A} = \mathscr{C}_2 = 0 \quad \text{and} \quad \mathscr{C}_1 = -1, \tag{46}$$

consistent with (37). For $n = 3$, we find $\mathscr{A}_3 = \mu(3 + 2v)/4$ and $\mathscr{C}_3 = (1 + v)/2$. The formulas for $n = 4$ can then be obtained from (B.23)–(B.28).

Similarly, for isotropic materials (for which $\hat{\alpha} = 1$), we find that

$$\hat{\mathscr{D}}_1 = -\mu, \quad \hat{\mathscr{D}}_2 = 0, \quad \hat{\mathscr{A}} = -5\mu/8 \quad \text{and} \quad \hat{\mathscr{C}} = -3/8, \tag{47}$$

consistent with (45).

6. Dispersion relation for rods

Having found solutions of the equation of motion, the next step is to apply the lateral boundary condition. We approximate this condition as (31); this vector equation has three components and we have three unknown constants, namely, u_0, $\tilde{v}_0$ and $\hat{w}_0$. Explicitly, (31) gives

$$\langle \mathbf{b}_0(\tilde{\alpha})\rangle_1 + \varepsilon\langle \mathbf{b}_1(\hat{\alpha})\rangle_1 + \varepsilon^2\langle \mathbf{b}_2(0)\rangle_1 + \varepsilon^2\langle \mathbf{b}_2(\tilde{\alpha})\rangle_1 + \varepsilon^4\langle \mathbf{b}_4(0)\rangle_1 = 0,$$
$$\langle \mathbf{b}_0(\tilde{\alpha})\rangle_2 + \varepsilon\langle \mathbf{b}_1(\hat{\alpha})\rangle_2 + \varepsilon^2\langle \mathbf{b}_2(0)\rangle_2 + \varepsilon^2\langle \mathbf{b}_2(\tilde{\alpha})\rangle_2 + \varepsilon^4\langle \mathbf{b}_4(0)\rangle_2 = 0,$$
$$\langle \mathbf{b}_0(\hat{\alpha})\rangle_3 + \varepsilon\langle \mathbf{b}_1(\tilde{\alpha})\rangle_3 + \varepsilon^2\langle \mathbf{b}_2(\hat{\alpha})\rangle_3 + \varepsilon^3\langle \mathbf{b}_3(0)\rangle_3 = 0,$$

once vanishing contributions have been removed. Substituting the appropriate expressions for $\langle \mathbf{b}_i(\alpha)\rangle_j$, as obtained in Section 5, we obtain

$$\mathbf{Z}(u_0, \tilde{v}_0, \hat{w}_0)^{\mathrm{T}} = \mathbf{0}, \tag{48}$$

where

$$\mathbf{Z} = \begin{pmatrix} Z_{11} & \beta\tilde{\mathscr{C}}_1 + \tilde{\mathscr{E}} - \kappa^2\tilde{\mathscr{A}}_1 & \mathrm{i}\kappa\hat{\mathscr{D}}_1 \\ Z_{21} & \beta\tilde{\mathscr{C}}_2 - \tilde{\mathscr{E}} - \kappa^2\tilde{\mathscr{A}}_2 & \mathrm{i}\kappa\hat{\mathscr{D}}_2 \\ \mathrm{i}\kappa(\beta\mathscr{C}_3 - \kappa^2\mathscr{A}_3) & \mathrm{i}\kappa\tilde{\mathscr{D}}_3 & \beta\hat{\mathscr{C}} + \hat{\alpha}C_{55} - \kappa^2\hat{\mathscr{A}} \end{pmatrix},$$

$$Z_{11} = \beta\mathscr{C}_1 - \kappa^2\mathscr{A} + \beta^2\mathscr{H}_1 - \beta\kappa^2\mathscr{J}_1 + \kappa^4\mathscr{L}_1,$$
$$Z_{21} = \beta\mathscr{C}_2 + \kappa^2\mathscr{A} + \beta^2\mathscr{H}_2 - \beta\kappa^2\mathscr{J}_2 + \kappa^4\mathscr{L}_2,$$

$\beta = \varrho(\omega a)^2$, a is the cross-sectional radius, and, in the notation of Achenbach (1973, Chapter 6), we have $\varepsilon\xi = ka = \kappa$, say; k is the axial wavenumber (so that the axial wavelength is $2\pi/k$). For non-trivial solutions, we require $\det\mathbf{Z} = 0$; this gives

$$Z_{11}\{(\beta\tilde{\mathscr{C}}_2 - \tilde{\mathscr{E}} - \kappa^2\tilde{\mathscr{A}}_2)(\beta\hat{\mathscr{C}} + \hat{\alpha}C_{55} - \kappa^2\hat{\mathscr{A}}) + \kappa^2\hat{\mathscr{D}}_2\tilde{\mathscr{D}}_3\} - Z_{21}\{(\beta\tilde{\mathscr{C}}_1 + \tilde{\mathscr{E}} - \kappa^2\tilde{\mathscr{A}}_1)(\beta\hat{\mathscr{C}} + \hat{\alpha}C_{55} - \kappa^2\hat{\mathscr{A}}) + \kappa^2\hat{\mathscr{D}}_1\tilde{\mathscr{D}}_3\} + \kappa^2(\beta\mathscr{C}_3 - \kappa^2\mathscr{A}_3)\{\hat{\mathscr{D}}_1(\beta\tilde{\mathscr{C}}_2 - \tilde{\mathscr{E}} - \kappa^2\tilde{\mathscr{A}}_2) - \hat{\mathscr{D}}_2(\beta\tilde{\mathscr{C}}_1 + \tilde{\mathscr{E}} - \kappa^2\tilde{\mathscr{A}}_1)\} = 0, \quad (49)$$

which involves κ as κ^2. Apart from β and κ, all the other coefficients in (49) are material constants. Eq. (49) is our main result: it is a dispersion relation governing the propagation of long flexural waves along a rod made from an elastic material with cylindrical orthotropy.

6.1. Long-wavelength approximations

We anticipate that our model is appropriate for long waves. Thus, guided by (49), we put

$$\beta = \kappa^2\Omega_2 + \kappa^4\Omega_4 + O(\kappa^6) \quad \text{as } \kappa \to 0, \quad (50)$$

and then determine Ω_2 and Ω_4. Using the general results from Appendix A, we have

$$\det\mathbf{Z} = \kappa^2\mathscr{Z}_2 + \kappa^4\mathscr{Z}_4 + \text{higher powers of } \kappa^2. \quad (51)$$

In the notation of (A.1)–(A.5), we have $A_2 = \mathscr{C}_1\Omega_2 - \mathscr{A}$, $D_2 = \mathscr{C}_2\Omega_2 + \mathscr{A}$, $B_0 = -E_0 = -\tilde{\mathscr{E}}$ and $I_0 = \hat{\alpha}C_{55}$, so that

$$\mathscr{Z}_2 = \hat{\alpha}C_{55}\tilde{\mathscr{E}}\Omega_2, \quad (52)$$

where we have used (37) and (A.6). As $\det\mathbf{Z} = 0$, we deduce from $\mathscr{Z}_2 = 0$ that $\Omega_2 = 0$, as no other quantity in (52) can vanish. Hence, (50) reduces to

$$\beta = \kappa^4\Omega_4 + O(\kappa^6) \quad \text{as } \kappa \to 0. \quad (53)$$

This is consistent with the known exact result for isotropic rods, as given in (2); see (Achenbach, 1973, p. 248):

$$\beta = \varrho(\omega a)^2 = \frac{\mu(3\lambda + 2\mu)}{4(\lambda + \mu)}\kappa^4 + O(\kappa^6) \quad \text{as } \kappa \to 0. \quad (54)$$

For $\mathscr{Z}_4$, we use (A.7). As $\Omega_2 = 0$, we obtain $A_2 = -D_2 = -\mathscr{A}$, $A_4 = \mathscr{C}_1\Omega_4 + \mathscr{L}_1$, $D_4 = \mathscr{C}_2\Omega_4 + \mathscr{L}_2$, $B_2 = -\tilde{\mathscr{A}}_1$, $E_2 = -\tilde{\mathscr{A}}_2$, $I_2 = -\hat{\mathscr{A}}$, $C_1 = \mathrm{i}\hat{\mathscr{D}}_1$, $F_1 = \mathrm{i}\hat{\mathscr{D}}_2$, $G_3 = -\mathrm{i}\mathscr{A}_3$ and $H_1 = \mathrm{i}\tilde{\mathscr{D}}_3$. Hence, using (37) and (A.7), we obtain

$$\mathscr{Z}_4 = \hat{\alpha}C_{55}\tilde{\mathscr{E}}(\Omega_4 - \mathscr{L}_1 - \mathscr{L}_2) + \mathscr{A}_3\tilde{\mathscr{E}}(\hat{\mathscr{D}}_1 + \hat{\mathscr{D}}_2) + \mathscr{A}\{\hat{\alpha}C_{55}(\tilde{\mathscr{A}}_1 + \tilde{\mathscr{A}}_2) - \tilde{\mathscr{D}}_3(\hat{\mathscr{D}}_1 + \hat{\mathscr{D}}_2)\}.$$

Then, from $\mathscr{Z}_4 = 0$, we obtain

$$\Omega_4 = \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{A}_3 - (\tilde{\mathscr{A}}_1 + \tilde{\mathscr{A}}_2 + \tilde{\mathscr{D}}_3)\mathscr{A}/\tilde{\mathscr{E}}, \quad (55)$$

after using (45). We conclude that the long-wave form of the dispersion relation is $\varrho(\omega a)^2 = \Omega_4(ka)^4$, where Ω_4 is defined by (55).

Let us give Ω_4 more explicitly. Using (B.25), (B.28), (B.21) and (B.22), we obtain

$$4(\mathscr{L}_1 + \mathscr{L}_2) = f_4^{(3)} + g_4^{(3)} + 4C_{13}U_3 = -U_2C_{55} - V_2C_{44} - U_3(C_{44} + 3C_{55}),$$

for the second equality, we used (B.17) and (B.18). Hence, (B.15) and (B.14) give

$$4(\mathscr{L}_1 + \mathscr{L}_2 + \mathscr{A}_3) = 3U_2C_{55} - V_2C_{44} + U_3(9C_{55} - C_{44}) = -U_2(2C_{13} + C_{23}) + V_2C_{23} + C_{33}, \tag{56}$$

where U_2 and V_2 are given by (B.10) and (B.11), respectively. Thus,

$$\Delta_2(\mathscr{L}_1 + \mathscr{L}_2 + \mathscr{A}_3) = -|C_3| + C_{66}\{C_{33}(3C_{11} + 2C_{12} - C_{22}) - (C_{13} + C_{23})(3C_{13} - C_{23})\}, \tag{57}$$

where Δ_2 is defined by (B.7) and

$$| C_3 | = \det \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}. \tag{58}$$

Using (B.33), (B.34), (B.37) and (B.38), explicit calculation gives

$$(\tilde{\alpha} + 2)(\tilde{\mathscr{A}}_1 + \tilde{\mathscr{A}}_2) = -C_{44}(1 + \tilde{W}_1) - C_{55}\tilde{U}_0 - (\tilde{\alpha} + 1)C_{55}\tilde{W}_1,$$

where $\tilde{U}_0$ and $\tilde{W}_1$ are given by (38) and (B.31), respectively. Then, (B.36) gives

$$(\tilde{\alpha} + 2)(\tilde{\mathscr{A}}_1 + \tilde{\mathscr{A}}_2 + \tilde{\mathscr{D}}_3) = (\tilde{\alpha} + 1)C_{55}[\tilde{U}_0 + (\tilde{\alpha} + 1)\tilde{W}_1] - C_{44}(1 + \tilde{W}_1) = C_{23}(1 - \tilde{U}_0) - \tilde{\alpha}C_{13}\tilde{U}_0.$$

The other terms in (55), namely $\mathscr{A}$ and $\tilde{\mathscr{E}}$, are given by (B.12), (B.35) and (B.39). Thus, we obtain

$$(\tilde{\mathscr{A}}_1 + \tilde{\mathscr{A}}_2 + \tilde{\mathscr{D}}_3)/\tilde{\mathscr{E}} = \frac{C_{23}(\tilde{\alpha}C_{11} - C_{12} - C_{66}) - C_{13}\{\tilde{\alpha}(C_{12} + C_{66}) - C_{22} - C_{66}\}}{(\tilde{\alpha} + 2)C_{66}\{C_{22} - C_{12} + \tilde{\alpha}(1 - \tilde{\alpha})C_{11}\}}. \tag{59}$$

Summarising, the long-wave form of the dispersion relation is $\varrho(\omega a)^2 = \Omega_4(ka)^4$, where Ω_4 is defined by (55), (57) and (59). These formulas hold for cylindrically orthotropic materials. For such materials, we note that the compliance s_{33} is given by

$$s_{33}^{-1} = |C_3|/(C_{11}C_{22} - C_{12}^2),$$

where $|C_3|$ is defined by (58) (and it occurs in (57)).

Now, let us specialise to tetragonal solids, defined by (B.44). Then, (46) gives $\mathscr{A} = 0$, and (B.46) gives U_2 and V_2. Then, from (55) and (56), we obtain

$$\Omega_4 = \frac{1}{4}C_{33} - \frac{1}{2}C_{13}^2/(C_{11} + C_{12}) = \frac{1}{4}s_{33}^{-1} = \frac{1}{4}E_3,$$

where E_3 is Young's modulus in the z-direction. Hence, we recover (5). Finally, specialising further to isotropic solids, we recover the well-known result (54). If one looks at the long-wavelength form of the solutions to (48), supposing that $u_0 = O(1)$ as $\kappa \to 0$, one finds that $\hat{w}_0 = O(\kappa^3), \tilde{v}_0 = O(\kappa^2)$ if $\mathscr{A} \neq 0$, and $\tilde{v}_0 = O(\kappa^4)$ if $\mathscr{A} = 0$ (which includes isotropy). Thus, u_0 gives the main contribution. As $v_0 = \mathrm{j}u_0$ (see (32)), we see that the main effect is uniform displacement in the x-direction (real part with respect to j) or in the y-direction (imaginary part with respect to j): this is what one expects for simple flexural motions.

7. Conclusions

We have described a systematic procedure for solving the problem of wave propagation in a rod of circular cross-section. The rod is made from an elastic solid with cylindrical anisotropy. Detailed results were given for flexural motion and cylindrical orthotropy (nine elastic stiffnesses).

The theory employs power series in r. Note that we do not limit ourselves to polynomials in r, and so we are not limited, in principle, to very long waves. Nevertheless, it turns out that the low-order truncations obtained work best for longer waves. In fact, the equation of motion is satisfied exactly but the lateral boundary condition is satisfied approximately because the relevant power series are truncated. The main issue is how to truncate properly: we propose to truncate so that all terms up to a certain power of ka are included, and this ensures that we obtain the correct long-wave behaviour. The same truncation strategy can be used for other problems involving non-cylindrical but axisymmetric, anisotropic bars. Moreover, as in (Boström, 2000), we could attain improved accuracy by retaining more terms in the truncation, but this would only be practicable by using software for symbolic manipulation.

Appendix A. Expanding a determinant

In general, for a matrix

$$\mathbf{Z} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \tag{A.1}$$

we have $\det \mathbf{Z} = A(EI - FH) - B(DI - FG) + C(DH - EG)$. Suppose now that $A, B, \ldots, I$ are functions of a small parameter κ, and we want to estimate $\det \mathbf{Z}$ for small κ. Specifically, suppose that

$$A = A_2\kappa^2 + A_4\kappa^4 + O(\kappa^6), \quad B = B_0 + B_2\kappa^2 + O(\kappa^4), \tag{A.2}$$

$$C = C_1\kappa + O(\kappa^3), \quad D = D_2\kappa^2 + D_4\kappa^4 + O(\kappa^6), \tag{A.3}$$

$$E = E_0 + E_2\kappa^2 + O(\kappa^4), \quad F = F_1\kappa + O(\kappa^3), \tag{A.4}$$

$$G = G_3\kappa^3 + O(\kappa^5), \quad H = H_1\kappa + O(\kappa^3), \quad I = I_0 + I_2\kappa^2 + O(\kappa^4) \tag{A.5}$$

as $\kappa \to 0$. Then, we find that $\det \mathbf{Z}$ is given by (51), where

$$\mathscr{Z}_2 = (A_2E_0 - B_0D_2)I_0, \tag{A.6}$$

$$\mathscr{Z}_4 = (A_4E_0 - B_0D_4)I_0 + A_2(E_0I_2 + E_2I_0 - F_1H_1) - D_2(B_0I_2 + B_2I_0 - C_1H_1) + G_3(B_0F_1 - C_1E_0). \tag{A.7}$$

Notice that all terms given in (A.2)–(A.5) are needed (and no others) if one wants complete expressions for $\mathscr{Z}_2$ and $\mathscr{Z}_4$. This observation motivates the truncation (31).

Appendix B. Frobenius solutions

Solution for $\alpha = 0$. Eq. (16) (with $m = 1$ and $\alpha = 0$) becomes

$$\begin{pmatrix} c & \mathrm{j}c & 0 \\ -\mathrm{j}c & c & 0 \\ 0 & 0 & C_{44} \end{pmatrix} \begin{pmatrix} u_0 \\ \mathrm{j}v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

where $c = C_{22} + C_{66}$. As $c \neq 0$ and $C_{44} \neq 0$, we obtain

$$\mathbf{u}_0 = (u_0, \mathrm{j}u_0, 0)^{\mathrm{T}}, \tag{B.1}$$

for some constant u_0, whence (27) gives $\mathbf{b}_0(0) = \mathbf{0}$.

Next, writing $\mathbf{u}_1 = (u_1, \mathrm{j}v_1, w_1)^{\mathrm{T}}$, (17) (with $m = 1$ and $\alpha = 0$) reduces to

$$\begin{pmatrix} C_{11} - C_{22} - C_{66} & \mathrm{j}(C_{12} - C_{22}) \\ \mathrm{j}(C_{12} + C_{22} + 2C_{66}) & -C_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ \mathrm{j}v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B.2}$$

and

$$(C_{44} - C_{55})(w_1 + \mathrm{i}\xi u_0) = 0, \tag{B.3}$$

where we have used (B.1). The determinant of the 2×2 system (B.2) is

$$C_{11}C_{66}(1 - \tilde{\alpha}^2) = C_{12}^2 - C_{11}C_{22} + C_{66}(2C_{12} - C_{22}), \tag{B.4}$$

we assume that $\tilde{\alpha} \neq 1$ whence $u_1 = v_1 = 0$. (For isotropic solids, (B.4) reduces to $-3\mu(\lambda + 2\mu) \neq 0$.) Then, (28)–(30) give

$$\mathbf{b}_1(0) = [0, 0, C_{55}(w_1 + \mathrm{i}\xi u_0)]^{\mathrm{T}}. \tag{B.5}$$

The conclusion drawn from (B.3) depends on C_{44} and C_{55}: if $C_{44} \neq C_{55}$, we obtain $w_1 = -\mathrm{i}\xi u_0$ and $\mathbf{b}_1(0) = \mathbf{0}$, whereas if $C_{44} = C_{55}$ (as for isotropic solids), w_1 remains undetermined.

The next step is to consider (18) with $n = 2$, $m = 1$ and $\alpha = 0$. Writing $\mathbf{u}_2 = (u_2,\ \mathrm{j}v_2, w_2)^{\mathrm{T}}$, we obtain

$$\begin{pmatrix} 4C_{11} - C_{22} - C_{66} & \mathrm{j}(2C_{12} - C_{22} + C_{66}) \\ \mathrm{j}(2C_{12} + C_{22} + 3C_{66}) & 3C_{66} - C_{22} \end{pmatrix} \begin{pmatrix} u_2 \\ \mathrm{j}v_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \tag{B.6}$$

and $(C_{44} - 4C_{55})w_2 = 0$; we assume that this last equation gives $w_2 = 0$. In (B.6), the vector on the right-hand side has components

$$f_2 = -\mathrm{i}\xi(2C_{13} - C_{23} + C_{55})w_1 + \xi^2 C_{55}u_0 - \varrho(\omega L)^2 u_0,$$
$$g_2 = \mathrm{j}\{-\mathrm{i}\xi(C_{23} + C_{44})w_1 + \xi^2 C_{44}u_0 - \varrho(\omega L)^2 u_0\}.$$

Also, the determinant of the 2×2 system (B.6) is

$$4C_{11}C_{66}(4 - \tilde{\alpha}^2) = 4\{C_{12}^2 - C_{11}C_{22} + C_{66}(3C_{11} + 2C_{12} - C_{22})\} = \Delta_2, \tag{B.7}$$

say, which vanishes when $\tilde{\alpha} = 2$. This exceptional case includes isotropy. From (28)–(30), we obtain

$$\mathbf{b}_2(0) = [(2C_{11} + C_{12})u_2 - C_{12}v_2 + \mathrm{i}\xi C_{13}w_1, \mathrm{j}C_{66}(u_2 + v_2), 0]^{\mathrm{T}}. \tag{B.8}$$

Now, consider the generic situation in which $C_{44} \neq C_{55}$ and $\Delta_2 \neq 0$. (This excludes isotropy, which we shall return to later; in fact, we will obtain results for isotropic media by a limiting procedure.) From (B.3), we obtain $w_1 = -\mathrm{i}\xi u_0$, whence $f_2 = -\{\xi^2(2C_{13} - C_{23}) + \varrho(\omega L)^2\}u_0$, $g_2 = -\ \mathrm{j}\{\xi^2 C_{23} + \varrho(\omega L)^2\}u_0$ and, from (B.5), $\mathbf{b}_1(0) = \mathbf{0}$. Then, as $\Delta_2 \neq 0$, (B.6) determines u_2 and v_2 in terms of u_0. Explicitly,

$$u_2 = \{X_2\varrho(\omega L)^2 - \xi^2 U_2\}u_0 \quad \text{and} \quad v_2 = \{Y_2\varrho(\omega L)^2 - \xi^2 V_2\}u_0, \tag{B.9}$$

where

$$\begin{aligned} \Delta_2 U_2 &= 2\{C_{12}C_{23} - C_{13}C_{22} + C_{66}(3C_{13} - C_{23})\}, \\ \Delta_2 X_2 &= 2(C_{22} - C_{12} - 2C_{66}), \end{aligned} \tag{B.10}$$

$$\begin{aligned} \varDelta_2 V_2 &= 2\{C_{12}C_{23} - C_{13}C_{22} + 2(C_{11}C_{23} - C_{12}C_{13}) - C_{66}(3C_{13} - C_{23})\}, \\ \varDelta_2 Y_2 &= 2(C_{12} + C_{22} - 2C_{11} + 2C_{66}). \end{aligned} \tag{B.11}$$

From (B.8), we find that $\mathbf{b}_2(0)$ is given by (33), where

$$\begin{aligned} \varDelta_2 \mathscr{A} &= 4C_{66}\{(C_{13} - C_{23})C_{12} + C_{13}C_{22} - C_{11}C_{23}\}, \\ \varDelta_2 \mathscr{C}_1 &= 4\{C_{11}C_{22} - C_{12}^2 - 2C_{66}(C_{11} + C_{12})\}, \\ \varDelta_2 \mathscr{C}_2 &= 4C_{66}(C_{22} - C_{11}). \end{aligned} \tag{B.12}$$

Explicit calculation, using (26) and (B.7), gives (37).
Continuing with $n = 3$, we find that $\mathbf{u}_3 = (0, 0, w_3)^{\mathrm{T}}$, where

$$w_3 = \mathrm{i}\xi\{X_3 \varrho(\omega L)^2 - \xi^2 U_3\}u_0, \tag{B.13}$$

$$\begin{aligned} (9C_{55} - C_{44})X_3 &= 1 - (2C_{13} + C_{23} + 3C_{55})X_2 + (C_{23} + C_{44})Y_2, \\ (9C_{55} - C_{44})U_3 &= C_{33} - (2C_{13} + C_{23} + 3C_{55})U_2 + (C_{23} + C_{44})V_2. \end{aligned} \tag{B.14}$$

Hence, $\mathbf{b}_3(0)$ is given by (34), where

$$\mathscr{C}_3 = C_{55}(3X_3 + X_2) \quad \text{and} \quad \mathscr{A}_3 = C_{55}(3U_3 + U_2). \tag{B.15}$$

For $n = 4$, we obtain $\mathbf{u}_4 = (u_4, \mathrm{j}v_4, 0)^{\mathrm{T}}$, where

$$\begin{pmatrix} 16C_{11} - C_{22} - C_{66} & \mathrm{j}(4C_{12} - C_{22} + 3C_{66}) \\ \mathrm{j}(4C_{12} + C_{22} + 5C_{66}) & 15C_{66} - C_{22} \end{pmatrix} \begin{pmatrix} u_4 \\ \mathrm{j}v_4 \end{pmatrix} = \begin{pmatrix} f_4 \\ g_4 \end{pmatrix}, \tag{B.16}$$

$$\begin{aligned} f_4 &= \{\varrho^2(\omega L)^4 f_4^{(1)} - \varrho(\omega L)^2 \xi^2 f_4^{(2)} + \xi^4 f_4^{(3)}\}u_0, \\ g_4 &= \mathrm{j}\{\varrho^2(\omega L)^4 g_4^{(1)} - \varrho(\omega L)^2 \xi^2 g_4^{(2)} + \xi^4 g_4^{(3)}\}u_0, \end{aligned}$$

$$\begin{aligned} f_4^{(1)} &= -X_2, \\ f_4^{(2)} &= -X_2C_{55} - U_2 - X_3(4C_{13} - C_{23} + 3C_{55}), \\ f_4^{(3)} &= -U_2C_{55} - U_3(4C_{13} - C_{23} + 3C_{55}), \end{aligned} \tag{B.17}$$

$$\begin{aligned} g_4^{(1)} &= -Y_2, \\ g_4^{(2)} &= -Y_2C_{44} - V_2 - X_3(C_{23} + C_{44}), \\ g_4^{(3)} &= -V_2C_{44} - U_3(C_{23} + C_{44}). \end{aligned} \tag{B.18}$$

Hence, we can write

$$u_4 = \{\varrho^2(\omega L)^4 X_4^{(1)} - \varrho(\omega L)^2 \xi^2 X_4^{(2)} + \xi^4 X_4^{(3)}\}u_0, \tag{B.19}$$

$$v_4 = \{\varrho^2(\omega L)^4 Y_4^{(1)} - \varrho(\omega L)^2 \xi^2 Y_4^{(2)} + \xi^4 Y_4^{(3)}\}u_0, \tag{B.20}$$

where $X_4^{(\ell)}$ and $Y_4^{(\ell)}$ $(\ell = 1, 2, 3)$ solve separate systems,

$$\begin{pmatrix} 16C_{11} - C_{22} - C_{66} & C_{22} - 4C_{12} - 3C_{66} \\ 4C_{12} + C_{22} + 5C_{66} & 15C_{66} - C_{22} \end{pmatrix} \begin{pmatrix} X_4^{(\ell)} \\ Y_4^{(\ell)} \end{pmatrix} = \begin{pmatrix} f_4^{(\ell)} \\ g_4^{(\ell)} \end{pmatrix}.$$

Thus,

$$\Delta_4 X_4^{(\ell)} = (15C_{66} - C_{22})f_4^{(\ell)} + (4C_{12} - C_{22} + 3C_{66})g_4^{(\ell)}, \tag{B.21}$$

$$\Delta_4 Y_4^{(\ell)} = -(4C_{12} + C_{22} + 5C_{66})f_4^{(\ell)} + (16C_{11} - C_{22} - C_{66})g_4^{(\ell)}, \tag{B.22}$$

where $\Delta_4 = 16\{C_{12}^2 - C_{11}C_{22} + C_{66}(15C_{11} + 2C_{12} - C_{22})\}$ and $\ell = 1, 2, 3$.

The corresponding tractions are given by (35) and (36) and $\langle \mathbf{b}_4(0)\rangle_3 = 0$, where

$$\mathscr{H}_1 = (4C_{11} + C_{12})X_4^{(1)} - C_{12}Y_4^{(1)}, \tag{B.23}$$

$$\mathscr{J}_1 = (4C_{11} + C_{12})X_4^{(2)} - C_{12}Y_4^{(2)} + C_{13}X_3, \tag{B.24}$$

$$\mathscr{L}_1 = (4C_{11} + C_{12})X_4^{(3)} - C_{12}Y_4^{(3)} + C_{13}U_3, \tag{B.25}$$

$$\mathscr{H}_2 = C_{66}\{X_4^{(1)} + 3Y_4^{(1)}\}, \tag{B.26}$$

$$\mathscr{J}_2 = C_{66}\{X_4^{(2)} + 3Y_4^{(2)}\}, \tag{B.27}$$

$$\mathscr{L}_2 = C_{66}\{X_4^{(3)} + 3Y_4^{(3)}\}. \tag{B.28}$$

Summary for $\alpha = 0$.

$\mathbf{u}_0 = (u_0, \mathrm{j}v_0, 0)^{\mathrm{T}}$ with $v_0 = u_0$,
$\mathbf{u}_1 = (0, 0, w_1)^{\mathrm{T}}$ with $w_1 = -\mathrm{i}\xi u_0$,
$\mathbf{u}_2 = (u_2, \mathrm{j}v_2, 0)^{\mathrm{T}}$ with u_2 and v_2 given(uniquely) by (B.9),
$\mathbf{u}_3 = (0, 0, w_3)^{\mathrm{T}}$ with w_3 given by (B.13),
$\mathbf{u}_4 = (u_4, \mathrm{j}v_4, 0)^{\mathrm{T}}$ with u_4 and v_4 given by (B.19) and (B.20).

Also, $\mathbf{b}_0(0) = \mathbf{b}_1(0) = \mathbf{0}$, and $\mathbf{b}_2(0)$, $b_3(0)$ and $\mathbf{b}_4(0)$ are given by (33)–(36) in terms of the single unknown constant u_0.

Solution for $\alpha = \tilde{\alpha}$. Writing $u_0 = (\tilde{u}_0, j\tilde{v}_0, \tilde{w}_0)^{\mathrm{T}}$, we find that $\tilde{w}_0 = 0$ and that $\tilde{u}_0$ and $\tilde{v}_0$ are related by $\langle \mathbf{G}_0(\tilde{\alpha})\mathbf{u}_0\rangle_1 = 0$ (or $\langle \mathbf{G}_0(\tilde{\alpha})\mathbf{u}_0\rangle_2 = 0$):

$$\tilde{u}_0 = \tilde{U}_0 \tilde{v}_0 \tag{B.29}$$

with $\tilde{U}_0$ given by (38). Writing $u_1 = (\tilde{u}_1, j\tilde{v}_1, \tilde{w}_1)^{\mathrm{T}}$, we find that $\tilde{u}_1 = \tilde{v}_1 = 0$, whereas the equation $\langle \mathbf{G}_0(\tilde{\alpha}+1)\mathbf{u}_1 + \mathrm{i}\xi \mathbf{A}_1(\tilde{\alpha})\mathbf{u}_0\rangle_3 = 0$ gives

$$\tilde{w}_1 = \mathrm{i}\xi \tilde{W}_1 \tilde{v}_0, \tag{B.30}$$

where

$$[C_{44} - (\tilde{\alpha}+1)^2 C_{55}]\tilde{W}_1 = [\tilde{\alpha}C_{13} + C_{23} + (\tilde{\alpha}+1)C_{55}]\tilde{U}_0 - C_{23} - C_{44}. \tag{B.31}$$

Then, writing $\mathbf{u}_2 = (\tilde{u}_2, j\tilde{v}_2, \tilde{w}_2)^{\mathrm{T}}$, we find that $\tilde{w}_2 = 0$, whereas $\tilde{u}_2$ and $\tilde{v}_2$ solve a 2×2 system,

$$[(\tilde{\alpha}+2)^2 C_{11} - C_{22} - C_{66}]\tilde{u}_2 - [(\tilde{\alpha}+2)(C_{12} + C_{66}) - C_{22} - C_{66}]\tilde{v}_2 = \tilde{f}_2,$$
$$[(\tilde{\alpha}+2)(C_{12} + C_{66}) + C_{22} + C_{66}]\tilde{u}_2 + \{[(\tilde{\alpha}+2)^2 - 1]C_{66} - C_{22}\}\tilde{v}_2 = \tilde{g}_2,$$

where

$$\tilde{f}_2 = -[(\tilde{\alpha}+2)C_{13} - C_{23} + (\tilde{\alpha}+1)C_{55}]\mathrm{i}\xi\tilde{w}_1 + \xi^2 C_{55}\tilde{u}_0 - \varrho(\omega L)^2\tilde{u}_0 = \{\xi^2\tilde{F}_2 - \varrho(\omega L)^2\tilde{U}_0\}\tilde{v}_0,$$

$$\tilde{g}_2 = -(C_{23} + C_{44})\mathrm{i}\xi\tilde{w}_1 + \xi^2 C_{44}\tilde{v}_0 - \varrho(\omega L)^2\tilde{v}_0 = \{\xi^2\tilde{G}_2 - \varrho(\omega L)^2\}\tilde{v}_0,$$

$$\tilde{F}_2 = [(\tilde{\alpha} + 2)C_{13} - C_{23} + (\tilde{\alpha} + 1)C_{55}]\tilde{W}_1 + C_{55}\tilde{U}_0,$$

$$\tilde{G}_2 = (C_{23} + C_{44})\tilde{W}_1 + C_{44}.$$

The determinant of the system for $\tilde{u}_2$ and $\tilde{v}_2$ is $4C_{11}C_{66}(\tilde{\alpha}+1)(\tilde{\alpha}+2)^2 = \tilde{\Delta}_2$, say, which does not vanish. Hence, solving for $\tilde{u}_2$ and $\tilde{v}_2$, we obtain

$$\tilde{u}_2 = \{\tilde{X}_2\varrho(\omega L)^2 - \xi^2\tilde{U}_2\}\tilde{v}_0 \quad \text{and} \quad \tilde{v}_2 = \{\tilde{Y}_2\varrho(\omega L)^2 - \xi^2\tilde{V}_2\}\tilde{v}_0, \tag{B.32}$$

where

$$\begin{aligned}\tilde{\Delta}_2\tilde{U}_2 = {} & (\tilde{\alpha}+2)\tilde{W}_1\{C_{13}C_{22} - C_{12}C_{23} + (\tilde{\alpha}+1)C_{66}[C_{23} - (\tilde{\alpha}+3)C_{13}]\} + (1+\tilde{W}_1)C_{44}\{C_{22} - (\tilde{\alpha}+2)C_{12} \\ & - (\tilde{\alpha}+1)C_{66}\} + [\tilde{U}_0 + (\tilde{\alpha}+1)\tilde{W}_1]C_{55}\{C_{22} - (\tilde{\alpha}+1)(\tilde{\alpha}+3)C_{66}\},\end{aligned} \tag{B.33}$$

$$\begin{aligned}\tilde{\Delta}_2\tilde{V}_2 = {} & (\tilde{\alpha}+2)\tilde{W}_1\{C_{13}C_{22} - C_{12}C_{23} + (\tilde{\alpha}+2)(C_{12}C_{13} - C_{11}C_{23}) + C_{66}[(\tilde{\alpha}+3)C_{13} - C_{23}]\} \\ & + (1+\tilde{W}_1)C_{44}\{C_{22} - (\tilde{\alpha}+2)^2C_{11} + C_{66}\} + [\tilde{U}_0 + (\tilde{\alpha}+1)\tilde{W}_1]C_{55}\{(\tilde{\alpha}+2)C_{12} \\ & + C_{22} + (\tilde{\alpha}+3)C_{66}\},\end{aligned} \tag{B.34}$$

$$\begin{aligned}\tilde{\Delta}_2\tilde{X}_2 &= (1+\tilde{U}_0)C_{22} - (\tilde{\alpha}+2)C_{12} - (\tilde{\alpha}+1)[(\tilde{\alpha}+3)\tilde{U}_0 + 1]C_{66}, \\ \tilde{\Delta}_2\tilde{Y}_2 &= (1+\tilde{U}_0)C_{22} + (\tilde{\alpha}+2)\tilde{U}_0C_{12} - (\tilde{\alpha}+2)^2C_{11} + [(\tilde{\alpha}+3)\tilde{U}_0 + 1]C_{66}.\end{aligned}$$

The corresponding tractions are given by

$$\mathbf{b}_0(\tilde{\alpha}) = [(\tilde{\alpha}C_{11} + C_{12})\tilde{u}_0 - C_{12}\tilde{v}_0, \mathrm{j}C_{66}\{\tilde{u}_0 + (\tilde{\alpha}-1)\tilde{v}_0\}, 0]^{\mathrm{T}} = [\tilde{\mathscr{E}}_1\tilde{v}_0, \mathrm{j}\tilde{\mathscr{E}}_2\tilde{v}_0, 0]^{\mathrm{T}},$$

$$\mathbf{b}_1(\tilde{\alpha}) = [0, 0, (\tilde{\alpha}+1)C_{55}\tilde{w}_1 + \mathrm{i}\xi C_{55}\tilde{u}_0]^{\mathrm{T}} = [0, 0, \mathrm{i}\xi\tilde{\mathscr{D}}_3\tilde{v}_0]^{\mathrm{T}},$$

$$\langle\mathbf{b}_2(\tilde{\alpha})\rangle_1 = [(\tilde{\alpha}+2)C_{11} + C_{12}]\tilde{u}_2 - C_{12}\tilde{v}_2 + \mathrm{i}\xi C_{13}\tilde{w}_1 = \{\tilde{\mathscr{C}}_1\varrho(\omega L)^2 - \xi^2\tilde{\mathscr{A}}_1\}\tilde{v}_0,$$

$$\langle\mathbf{b}_2(\tilde{\alpha})\rangle_2 = \mathrm{j}C_{66}[\tilde{u}_2 + (\tilde{\alpha}+1)\tilde{v}_2] = \mathrm{j}\{\tilde{\mathscr{C}}_2\varrho(\omega L)^2 - \xi^2\tilde{\mathscr{A}}_2\}\tilde{v}_0$$

and $\langle\mathbf{b}_2(\tilde{\alpha})\rangle_3 = 0$, where

$$\tilde{\mathscr{E}}_1 = \tilde{\alpha}\tilde{U}_0C_{11} + (\tilde{U}_0 - 1)C_{12}, \quad \tilde{\mathscr{E}}_2 = (\tilde{\alpha} - 1 + \tilde{U}_0)C_{66}, \tag{B.35}$$

$$\tilde{\mathscr{D}}_3 = \{\tilde{U}_0 + (\tilde{\alpha}+1)\tilde{W}_1\}C_{55}, \tag{B.36}$$

$$\begin{aligned}\tilde{\mathscr{A}}_1 &= [(\tilde{\alpha}+2)C_{11} + C_{12}]\tilde{U}_2 - C_{12}\tilde{V}_2 + C_{13}\tilde{W}_1, \\ \tilde{\mathscr{C}}_1 &= [(\tilde{\alpha}+2)C_{11} + C_{12}]\tilde{X}_2 - C_{12}\tilde{Y}_2,\end{aligned} \tag{B.37}$$

$$\begin{aligned}\tilde{\mathscr{A}}_2 &= C_{66}[\tilde{U}_2 + (\tilde{\alpha}+1)\tilde{V}_2], \\ \tilde{\mathscr{C}}_2 &= C_{66}[\tilde{X}_2 + (\tilde{\alpha}+1)\tilde{Y}_2].\end{aligned} \tag{B.38}$$

Direct calculation, using (26) and (38), shows that $\tilde{\mathscr{E}}_1 + \tilde{\mathscr{E}}_2 = 0$, so that we can write

$$\tilde{\mathscr{E}}_1 \equiv \tilde{\mathscr{E}} \quad \text{and} \quad \tilde{\mathscr{E}}_2 = -\tilde{\mathscr{E}}. \tag{B.39}$$

Summary for $\alpha = \tilde{\alpha}$.

$$\mathbf{u}_0 = (\tilde{u}_0, \mathrm{j}\tilde{v}_0, 0)^{\mathrm{T}} \quad \text{with } \tilde{u}_0 \text{ and } \tilde{v}_0 \text{ related by (B.29)},$$
$$\mathbf{u}_1 = (0, 0, \tilde{w}_1)^{\mathrm{T}} \quad \text{with } \tilde{w}_1 \text{ given by (B.30)},$$
$$\mathbf{u}_2 = (\tilde{u}_2, \mathrm{j}\tilde{v}_2, 0)^{\mathrm{T}} \quad \text{with } \tilde{u}_2 \text{ and } \tilde{v}_2 \text{ given uniquely by (B.32)}.$$

We also found expressions for $\mathbf{b}_0(\tilde{\alpha})$, $\mathbf{b}_1(\tilde{\alpha})$ and $\mathbf{b}_2(\tilde{\alpha})$; they are given by (39)–(41), in terms of the single unknown constant $\tilde{v}_0$.

Solution for $\alpha = \hat{\alpha}$. Writing $\mathbf{u}_0 = (\hat{u}_0, \mathrm{j}\hat{v}_0, \hat{w}_0)^{\mathrm{T}}$, we find that $\hat{u}_0 = \hat{v}_0 = 0$. Next, writing $\mathbf{u}_1 = (\hat{u}_1, \mathrm{j}\hat{v}_1, \hat{w}_1)^{\mathrm{T}}$, (17) (with $m = 1$ and $\alpha = \hat{\alpha}$) gives $\hat{w}_1 = 0$ and

$$[(\hat{\alpha}+1)^2 C_{11} - C_{22} - C_{66})]\hat{u}_1 - [(\hat{\alpha}+1)C_{12} - C_{22} + \hat{\alpha}C_{66}]\hat{v}_1 = f_1, \tag{B.40}$$

$$[(\hat{\alpha}+1)(C_{12} + C_{66}) + C_{22} + C_{66}]\hat{u}_1 + [\hat{\alpha}(\hat{\alpha}+2)C_{66} - C_{22}]\hat{v}_1 = g_1, \tag{B.41}$$

where $f_1 = -\mathrm{i}\xi[(\hat{\alpha}+1)C_{13} - C_{23} + \hat{\alpha}C_{55}]\hat{w}_0$ and $g_1 = -\mathrm{i}\xi(C_{23} + C_{44})\hat{w}_0$. The determinant of the 2×2 system (B.40) and (B.41) is

$$C_{11}C_{66}(\hat{\alpha}+1)^2\{(\hat{\alpha}+1)^2 - \tilde{\alpha}^2\} = (\hat{\alpha}+1)\hat{\Delta}_1,$$

say; this vanishes when $\hat{\alpha} = \tilde{\alpha} - 1$, as for isotropic solids. Assuming that $\hat{\Delta}_1 \neq 0$, we can solve (B.40) and (B.41) uniquely for $\hat{u}_1$ and $\hat{v}_1$ in terms of $\hat{w}_0$. Explicitly, we obtain

$$\hat{u}_1 = \mathrm{i}\xi U_1 \hat{w}_0 \quad \text{and} \quad \hat{v}_1 = \mathrm{i}\xi V_1 \hat{w}_0, \tag{B.42}$$

where

$$\hat{\Delta}_1 U_1 = -\hat{\alpha}C_{66}[(\hat{\alpha}+2)C_{13} - C_{23} + 2\hat{\alpha}C_{55}] + \hat{\alpha}C_{55}(C_{22} - \hat{\alpha}C_{12}) + C_{13}C_{22} - C_{12}C_{23},$$

$$\hat{\Delta}_1 V_1 = C_{66}[(\hat{\alpha}+2)C_{13} - C_{23} + 2\hat{\alpha}C_{55}] + \hat{\alpha}C_{55}[C_{12} + C_{22} - \hat{\alpha}(\hat{\alpha}+1)C_{11}] + (\hat{\alpha}+1)(C_{12}C_{13} - C_{11}C_{23}) + C_{13}C_{22} - C_{12}C_{23}.$$

Next, writing $\mathbf{u}_2 = (\hat{u}_2, \mathrm{j}\hat{v}_2, \hat{w}_2)^{\mathrm{T}}$, we obtain $\hat{u}_2 = \hat{v}_2 = 0$ and

$$\begin{aligned} 4(\hat{\alpha}+1)C_{55}\hat{w}_2 &= -\mathrm{i}\xi[(\hat{\alpha}+1)C_{13} + C_{23} + (\hat{\alpha}+2)C_{55}]\hat{u}_1 + \mathrm{i}\xi(C_{23} + C_{44})\hat{v}_1 + \xi^2 C_{33}\hat{w}_0 - \varrho(\omega L)^2\hat{w}_0 \\ &= -\{\varrho(\omega L)^2 + \xi^2 W_2\}\hat{w}_0, \end{aligned} \tag{B.43}$$

where $W_2 = -U_1[(\hat{\alpha}+1)C_{13} + C_{23} + (\hat{\alpha}+2)C_{55}] + V_1(C_{23} + C_{44}) - C_{33}$.

The corresponding tractions are given by

$$\mathbf{b}_0(\hat{\alpha}) = [0, 0, \hat{\alpha}C_{55}\hat{w}_0]^{\mathrm{T}},$$
$$\langle \mathbf{b}_1(\hat{\alpha}) \rangle_1 = [(\hat{\alpha}+1)C_{11} + C_{12}]\hat{u}_1 - C_{12}\hat{v}_1 + \mathrm{i}\xi C_{13}\hat{w}_0 = \mathrm{i}\xi\hat{\mathscr{D}}_1\hat{w}_0,$$
$$\langle \mathbf{b}_1(\hat{\alpha}) \rangle_2 = \mathrm{j}C_{66}(\hat{u}_1 + \hat{\alpha}\hat{v}_1) = \mathrm{j}\mathrm{i}\xi\hat{\mathscr{D}}_2\hat{w}_0,$$
$$\langle \mathbf{b}_2(\hat{\alpha}) \rangle_3 = (\hat{\alpha}+2)C_{55}\hat{w}_2 + \mathrm{i}\xi C_{55}\hat{u}_1 = \{\hat{\mathscr{C}}\varrho(\omega L)^2 - \xi^2\hat{\mathscr{A}}\}\hat{w}_0,$$

and $\langle \mathbf{b}_1(\hat{\alpha}) \rangle_3 = \langle \mathbf{b}_2(\hat{\alpha}) \rangle_1 = \langle \mathbf{b}_2(\hat{\alpha}) \rangle_2 = 0$, where

$$\hat{\mathscr{D}}_1 = [(\hat{\alpha}+1)C_{11} + C_{12}]U_1 - C_{12}V_1 + C_{13},$$
$$\hat{\mathscr{D}}_2 = C_{66}(U_1 + \hat{\alpha}V_1),$$
$$\hat{\mathscr{A}} = \frac{1}{4}[(\hat{\alpha}+2)/(\hat{\alpha}+1)]W_2 + C_{55}U_1,$$
$$\hat{\mathscr{C}} = -\frac{1}{4}[(\hat{\alpha}+2)/(\hat{\alpha}+1)].$$

Explicit calculations reveal that $\hat{\mathscr{D}}_1$ and $\hat{\mathscr{D}}_2$ are related by (45).

Summary for $\alpha = \hat{\alpha}$.

$$\mathbf{u}_0 = (0, 0, \hat{w}_0)^{\mathrm{T}},$$
$$\mathbf{u}_1 = (\hat{u}_1, \mathrm{j}\hat{v}_1, 0)^{\mathrm{T}} \quad \text{with } \hat{u}_1 \text{ and } \hat{v}_1 \text{ given uniquely by (B.42)},$$
$$\mathbf{u}_2 = (0, 0, \hat{w}_2)^{\mathrm{T}} \quad \text{with } \hat{w}_2 \text{ given by (B.43)}.$$

We also found expressions for $\mathbf{b}_0(\hat{\alpha})$, $\mathbf{b}_1(\hat{\alpha})$ and $\mathbf{b}_2(\hat{\alpha})$; they are given by (42)–(44) in terms of the single unknown constant $\hat{w}_0$.

Tetragonal materials. The formulas given above for $\alpha = 0$ and for $\alpha = \hat{\alpha}$ are degenerate for isotropic materials. However, the relevant formulas are easily obtained by first specialising to tetragonal materials, for which

$$C_{11} = C_{22}, \quad C_{13} = C_{23} \quad \text{and} \quad C_{44} = C_{55}, \tag{B.44}$$

see (Ting, 1996, p. 46). Formulas for isotropy and for transverse isotropy are then obtained by simple substitution. Note that the formulas for $\alpha = \tilde{\alpha}$ are not degenerate.

Beginning with $\alpha = 0$, we make use of (B.44) and find that

$$\Delta_2 = 4(C_{11} + C_{12})(C_{12} + 2C_{66} - C_{11}), \tag{B.45}$$

$$U_2 = -V_2 = \frac{1}{2} C_{13}/(C_{11} + C_{12}), \tag{B.46}$$

$$X_2 = -Y_2 = -\frac{1}{2}\Big/(C_{11} + C_{12}).$$

Also, in (33), we obtain (46). Then, for $n = 3$, we find

$$U_3 = \frac{C_{33}(C_{11} + C_{12}) - 2C_{13}(C_{13} + C_{55})}{8C_{55}(C_{11} + C_{12})},$$
$$X_3 = \frac{C_{11} + C_{12} + 2(C_{13} + C_{55})}{8C_{55}(C_{11} + C_{12})},$$
$$\mathscr{A}_3 = \frac{3C_{33}(C_{11} + C_{12}) - 2C_{13}(3C_{13} + C_{55})}{8(C_{11} + C_{12})},$$
$$\mathscr{C}_3 = \frac{3(C_{11} + C_{12} + 2C_{13}) + 2C_{55}}{8(C_{11} + C_{12})}.$$

The formulas for $n = 4$ can then be obtained by evaluating (B.23)–(B.28).

Similarly, for $\alpha = \hat{\alpha}$, we find that $\hat{\Delta}_1 = 2(C_{11} + C_{12})(C_{12} + 2C_{66} - C_{11})$,

$$U_1 = -V_1 = -\frac{1}{2}(C_{13} + C_{55})/(C_{11} + C_{12}),$$
$$\hat{\mathscr{D}}_1 = -C_{55}, \quad \hat{\mathscr{D}}_2 = 0, \quad \hat{\mathscr{C}} = -3/8,$$
$$\hat{\mathscr{A}} = \frac{2(C_{13} + C_{55})(3C_{13} + C_{55}) - 3C_{33}(C_{11} + C_{12})}{8(C_{11} + C_{12})}.$$

The formulas for $\hat{\mathscr{D}}_1$ and $\hat{\mathscr{D}}_2$ are consistent with (45).

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